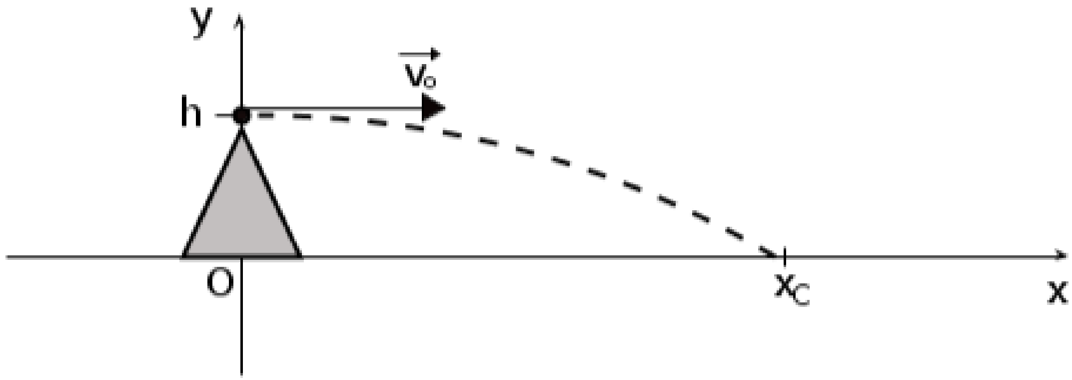
1 Problem

A bullet of mass m is fired from the top of a mountain of height h with initial velocity $\vec{v_0} = v_0\hat{x}$, which is parallel to the ground. After a time t_c it hits the ground at a distance x_c from the mountain. Calculate t_c and x_c. (I used the letter c as the 'time of crash' and the 'x-coordinate of crash').



2 Suggestions

Remember that any vector $\vec{u}$ can be expressed as

$$\vec{u} = u\hat{u} \tag{1}$$

and also as

$$\vec{u} = u_x\hat{x} + u_y\hat{y} \tag{2}$$

3 Solution

Newton tells us that a body of mass m subjected to a force $\vec{F}$ (or to several forces whose sum is $\vec{F}$) has an acceleration $\vec{a}$ and these 3 quantities are related by this law:

$$\vec{F} = m\vec{a} \tag{3}$$

But how many forces is our bullet subjected to, right after it is fired from the top of the mountain? Only one: the gravitational force.

$$\vec{F} = m\vec{g} \tag{4}$$

Then putting (4) into (3) we obtain

$$m\vec{a} = m\vec{g} \quad \Rightarrow \quad \vec{a} = \vec{g} \tag{5}$$

The vector $\vec{g}$ is the gravitational acceleration, and (as every vector) can be written as

$$\vec{g} = g\hat{g} \tag{6}$$

Where $g = 9.81\frac{m}{s^2}$ is a value (almost) constant on the surface of our planet, and $\hat{g}$ is the versor pointing down to the center of our planet. If it points down, then it has the opposite sense of our chosen y axis, which points up, as you can see from the picture. So we can say

$$\hat{g} = -\hat{y} \tag{7}$$

So from (6) and (7) we see that

$$\vec{a} = -g\hat{y} \tag{8}$$

We know that, as every vector, $\vec{a}$ can be written as $\vec{a} = a_x\hat{x} + a_y\hat{y}$, so from (8) we see that

$$a_x = 0 \qquad \text{and} \qquad a_y = -g \tag{9}$$

We found that there is no acceleration along the x direction, while there's a negative constant acceleration in the y direction. This happens because the only force present acts only vertically. These considerations are valid along all the trajectory of the bullet from the mountain to the ground, because the force acting is always the same, it doesn't change.

3.1 motion in the x direction

So now let's see what happens along the x direction. We found that there is no acceleration. A motion with no acceleration is called *uniform* and follows this law:

$$x(t) = x_0 + v_{0x}(t - t_0) \tag{10}$$

Where x_0 is the initial x-coordinate of the bullet, so from the figure we see that $x_0 = 0$. We also choose to start our clock exactly when the bullet is fired, so the initial time is $t_0 = 0$.
v_{0x} is the x-component of the initial velocity. We can write as always

$$\vec{v_0} = v_{0x}\hat{x} + v_{0y}\hat{y} \tag{11}$$

but the problem says $\vec{v_0} = v_0\hat{x}$, from which we find

$$v_{0x} = v_0 \qquad \text{and} \qquad v_{0y} = 0 \tag{12}$$

We can now rewrite (10) as

$$x(t) = v_0 t \tag{13}$$

3.2 motion in the y direction

As said, we found that there is a constant acceleration in the y direction. A motion with constant acceleration is called *uniformly accelerated* and follows this law:

$$y(t) = y_0 + v_{0y}(t - t_0) + \frac{1}{2}a_y(t - t_0)^2 \tag{14}$$

The initial y-coordinate of the bullet is equal to the height of the mountain, $y_0 = h$, as you can see from the figure. Then using also (12) and (9) we rewrite (14) as

$$y(t) = h - \frac{1}{2}gt^2 \tag{15}$$

3.3 When the bullet hits the ground

From the figure we see that when the bullet reaches the ground the y-coordinate is 0, and we called that time t_c, so we say $y(t_c) = 0$, then from (15) we have

$$h - \frac{1}{2}gt_c^2 = 0 \qquad \Rightarrow \qquad t_c = \pm\sqrt{\frac{2h}{g}} \tag{16}$$

We will neglect the negative solution, we only care about events following our initial time $t_0 = 0$. So the solution is

$$t_c = \sqrt{\frac{2h}{g}} \tag{17}$$

Now to find x_c we simply have to think that it is the x-coordinate at the time of crash, so $x_c = x(t_c)$. From (13) we have

$$x_c = v_0 t_c \tag{18}$$

so, using (17) the solution is

$$x_c = v_0\sqrt{\frac{2h}{g}} \tag{19}$$

3.4 Trajectory

We write (13) and (15) together and call it *parametric equation* of the trajectory

$$\begin{cases} x(t) = & v_0 t \\ y(t) = & h - \frac{1}{2}gt^2 \end{cases} \qquad t \in [0, t_c] \tag{20}$$

From the first line of (20) we find $t = \frac{x}{v_0}$, which we put into the second line to find the *cartesian equation* of the trajectory

$$y(x) = -\left(\frac{g}{2v_0^2}\right)x^2 + h \qquad\qquad x \in [0, x_c] \tag{21}$$

from which you can recognise the equation of a piece of parabola.
A bullet fired horizontally from the top of a mountain traces a piece of parabola.

What happens if the bullet is not fired horizontally? In other words, what happens if $v_0y \neq 0$?